FMN011 — Seminar 2 — Solution of Linear Systems

1. True or false:

   (a) If A is nonsingular, then the number of solutions to $Ax = b$ depends on the particular choice of vector b.

   (b) For a symmetric matrix S, it is always the case that $\|S\|_1 = \|S\|_\infty$.

   (c) If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.

   (d) The product of two upper triangular matrices is also upper triangular.

   (e) If the rows of a square matrix are linearly dependent, then the columns of the matrix are also linearly dependent.

   (f) If A is any $n \times n$ matrix and P is any $n \times n$ permutation matrix, then $PA = AP$.

   (g) For $x \in \mathcal{R}^n$, $\|x\|_1 \geq \|x\|_\infty$.

   (h) If $\det(A) = 0$, then $\|A\| = 0$.

   (i) The product of two symmetric matrices is also symmetric.

   (j) $\kappa_p(A) = \kappa_p(A^{-1})$.

   (k) A system $Ax = b$ can have exactly two distinct solutions.

   (l) Every nonsingular matrix A can be written as $A = LU$, where L is lower triangular and U is upper triangular.

   (m) A singular matrix does not have an LU(P) factorization.

2. Given $Ax = b$, what effect on the solution vector x results from

   (a) Permuting the rows of $[A \quad b]$?

   (b) Permuting the columns of A?

   (c) Multiplying both sides of the equation from the left by a nonsingular matrix M?

3. Consider the matrix

$$A = \begin{bmatrix} 4 & -8 & 1 & 2 \\ 6 & 5 & 7 & 3 \\ 0 & -10 & -3 & 5 \\ 5 & -1 & 1 & 0 \end{bmatrix}$$

   What will the initial pivot in Gaussian elimination be if

   (a) No pivoting is used?

   (b) Partial pivoting is used?

4. Given $n \times n$ matrices A and B, what is the best way to compute $A^{-1}B$?

5. If x is a column vector and A is a matrix, which of the following computations require less work?

   (a) $y = (xx^T)A$

   (b) $y = x(x^T A)$

6. What is the inverse of a permutation matrix P?

7. Assume you have already computed the LU factorization, $PA = LU$. How would you use it to solve the system $A^T x = b$?

8. Classify each matrix as well conditioned or ill conditioned:

   (a) $\begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{pmatrix}$

   (b) $\begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{10} \end{pmatrix}$

   (c) $\begin{pmatrix} 10^{-10} & 0 \\ 0 & 10^{-10} \end{pmatrix}$

   (d) $\begin{pmatrix} 1.0000001 & 2 \\ 2 & 4 \end{pmatrix}$

9. In solving a linear system $Ax = b$, what is meant by the *residual* of an approximate solution $\hat{x}$? Does a small relative residual always imply that the solution is accurate?

10. Rank the following methods according to the amount of work required for solving most systems:

    (a) Gaussian elimination with partial pivoting

    (b) Explicit matrix inversion followed by matrix–vector multiplication

11. One of the following three systems could not be solved by Gauss-Seidel. Which one is it?

$$\begin{aligned} -3x_1 - 6x_2 + 2x_3 &= -61.5 \\ x_1 + x_2 + 5x_3 &= -21.5 \\ 10x_1 + 2x_2 - x_3 &= 27 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 7 \\ x_1 + 4x_2 - x_3 &= 4 \\ 3x_1 + x_2 - x_3 &= 3 \end{aligned}$$

$$\begin{aligned} -x_1 + 3x_2 + 5x_3 &= 7 \\ -2x_1 + 4x_2 - 5x_3 &= -3 \\ 2x_2 - x_3 &= 1 \end{aligned}$$

12. Suppose we have an iterative method to solve $Ax = b$,

$$x_{k+1} = x_k + L^{-1} r_k$$

where L is a very large sparse lower triangular matrix, A is a sparse matrix, and $r_k = Ax_k - b$. What is a good way to compute x_{k+1}? What is the order of the number of computations that must be performed at each step?