Numerical Analysis — FMN011 — 140602
Solutions

The exam lasts 4 hours and has 14 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. I have designed a new iterative method for solving non-linear equations, and I want to test it by solving the equation

$$4x^3 + x - 1 = 0,$$

whose exact solution is $x = 0.5$. When I apply the method with initial guess $x_0 = 1$, I get the following sequence: $\{1, 0.1464, 0.2897, 0.4036, 0.4701, 0.4948\}$.

   (a) **(2p)** Calculate the absolute error at each iteration step
   Solution: $x - x_k = [0.5000, -0.3536, -0.2103, -0.0964, -0.0299, -0.0052]$

   (b) **(2p)** Linear convergence is defined by $\|e_{k+1}\| \leq c \cdot \|e_k\|$ and $0 < c < 1$. Does the method converge linearly?
   Solution:

iteration	error	quotient
1	0.5	
2	-0.3536	0.7072
3	-0.2103	0.5947
4	-0.0964	0.4584
5	-0.0299	0.3102
6	-0.0052	0.1739

   $\|e_{k+1}\|/\|e_k\|$ is not constant, so the convergence is not linear.

2. **(6p)** Construct a fixed point iteration that will solve $x^4 + 0.5x^2 - x + 0.25 = 0$. Can it converge to the solution $x = 0.3053$? Can it converge to the solution $x = 0.6630$? Explain.

Solution:

$$\begin{aligned}
0 &= x^4 + 0.5x^2 - x + 0.25 \\
x &= x^4 + 0.5x^2 + 0.25 \\
x_{k+1} &= x_k^4 + 0.5x_k^2 + 0.25
\end{aligned}$$

$g(x) = x^4 + 0.5x^2 + 0.25 \Rightarrow g'(x) = 4x^3 + x$.
$|g'(0.3053)| = 0.4191 < 1$, so it will converge to 0.3053 for x_0 close enough to the solution.
$|g'(0.6630)| = 1.8287 > 1$, so it will not converge to 0.6630.

3. Construct an iterative method for solving $Ax = b$, $x_{k+1} = Bx_k + c$, by partitioning A as $A = M - N$ and letting $M = I$.
   (a) **(3p)** What is the iteration matrix, B? What is c?
   Solution: $A = I - N \Rightarrow N = I - A$. $(I - N)x = b \Rightarrow x = (I - A)x + b$. The iteration matrix is $B = I - A$ and $c = b$.
   (b) **(2p)** Give a good reason why we can solve the system

   $$\begin{aligned}
   0.5x - 0.4y &= 1 \\
   -0.1x + 1.3y &= 2
   \end{aligned}$$

   with this method.
   Solution: $I - A = \begin{pmatrix} 0.5 & 0.4 \\ 0.1 & -0.3 \end{pmatrix}$ and $\|I - A\|_1 = 0.7 < 1 \Rightarrow$ the method converges.

4. **(5p)** I read on the web that the following statements are true, but they are actually false. Give a good argument to show their falseness.
   (a) Every matrix has an LU factorization, $A = LU$.
   Solution: Every square matrix has a factorization $PA = LU$.
   (b) If a matrix A is not invertible, and $A = LU$, then L is not invertible.
   Solution: L is triangular with ones on the diagonal, so it is invertible.
   (c) After solving $Ax = b$, a small residual implies a small error.
   Solution: Only if A is well-conditioned.
   (d) Solving a linear system by means of LU factorization is numerically more stable than using Gauss elimination.
   Solution: They do the same arithmetic operations.
   (e) Partial pivoting may change the 1-norm condition number of a matrix.
   Solution: Partial pivoting is equivalent to exchanging rows, so the sum of the elements of each row remains the same.

5. **(5p)** Consider

$$A = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   (a) Show that the columns of A are orthonormal vectors.
   Solution: $1/\sqrt{2} \cdot 2/3 + 1/\sqrt{2} \cdot (-2/3) + 0 \cdot 1/3 = 0$, and $(1/\sqrt{2})^2 + (1/\sqrt{2})^2 + 0^2 = 1$, $(2/3)^2 + (-2/3)^2 + (1/3)^2 = 1$.

(b) How can (a) be used to solve the least squares problem $Ax = b$ efficiently?
Solution: $A^{\mathrm{T}}A = I$, so the normal equations become a matrix-vector multiplication, $x = A^{\mathrm{T}}b$.

(c) Calculate the least squares solution.

Solution: $x = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 1/3 \end{pmatrix}$

(d) Calculate the residual.

Solution: $r = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 1/3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/9 \\ -2/9 \\ -8/9 \end{pmatrix}$

(e) Is there a vector v such that $\|Av - b\|_2 < 2\sqrt{2}/3$?
Solution: No, because $\|r\|_2 = 2\sqrt{2}/3$ is the minimum possible 2-norm of the residual.

6. **(5p)** A friend tells me that

$$A = \begin{pmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix}$$

has three different eigenvalues, and that its corresponding eigenvectors are $(2/3, 2/3, -1/3)^{\mathrm{T}}, (-1/3, 2/3, 2/3)^{\mathrm{T}}$ and $(1, 0, 0)^{\mathrm{T}}$. Without calculating the eigenvectors, I tell her that she is wrong: it looks like one of the eigenvectors was incorrectly calculated. Can you reason how I concluded that?
Solution: As A is symmetric with distinct eigenvalues, its eigenvectors must the orthogonal, but $(1, 0, 0)^{\mathrm{T}}$ is not orthogonal to the other two.

7. **(5p)** When certain exact data,

x	1	2.15	3.2	5.1	5.4
y	0.100000	0.9938375	3.2768	13.2651	15.7464

is plotted on a loglog scale, (i.e., the values of $\log(x_i)$ and $\log(y_i)$ are plotted), the outcome looks like a straight line. What is the type of relation between x and y? Write the equation of the model. What would the parameters of the model be? Describe how to find them; you do not need to do the computations.
Solution: As $\log(y) = m\log(x) + b \Rightarrow \log(y) - \log(x^m) = b \Rightarrow \log(y/x^m) = b$ or $y = e^b x^m = kx^m$ (letting $k = e^b$). The parameters of the model are k and m. To find them, we can find the least squares solution of $\log(y) = m\log(x) + b$. As the data is exact, we can also take any two values from the table, and substitute them in the equation $y = kx^m$.

8. **(5p)** A quadratic Bézier curve that starts at (0,0) with a slope of 1/2, and ends at (1,1) with a slope of -2/3. Make a sketch of the curve, give all of its control points, and show the control polygon.
Solution: As the curve is quadratic, its has three control points. Let (x, y)

be the missing (intermediate) control point. Then $\frac{y-0}{x-0} = \frac{1}{2}$ and $\frac{y-1}{x-1} = -\frac{2}{3}$. Solving these two equations gives the point $(10/7, 5/7)$.

9. **(5p)** Explain why: if an $n \times n$ matrix A has eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, the power method will converge (for any appropriate choice of initial vector).
Solution: As A has n distinct eigenvalues, it has a set of n linearly independent eigenvectors. As all eigenvalues are positive, λ_1 is the only eigenvector with maximum modulus. These two conditions imply the power method will converge.

10. **(5p)** The DFT of a real vector is

```
-0.3536
-0.3536 + 3.5990i
-0.3536 - 0.9239i
-0.3536 - 0.2294i
-0.3536
-0.3536 + 0.2294i
-0.3536 + 0.9239i
-0.3536 - 3.5990i
```

Given that the DFT trigonometric interpolation polynomial is

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} (a_k \cos(2\pi kt) - b_k \sin(2\pi kt)) + \frac{a_{n/2}}{\sqrt{n}} \cos(n\pi t),$$

construct a low-pass filter with frequencies up to $2\pi t$.
Solution:

$$P(t) = \frac{-0.3536}{\sqrt{8}} + \frac{2}{\sqrt{8}}(-0.3536\cos(2\pi t) - 3.5990\sin(2\pi t)))$$

11. Consider the message

SKÅNSKT KNÄCKE

(a) **(1p)** Construct a Huffman tree for this message

(b) **(2p)** Construct a table with the binary code for each symbol.
Solution:

	S	K	Å	N	T	⊔	Ä	C	E
p_i	2	4	1	2	1	1	1	1	1
code	110	00	101	111	100	0111	0110	0101	0100

(c) **(1p)** How many bits are required to code the message?
Solution: 42

(d) **(1p)** What is the average number of bits/symbol used?
Solution: 42/14=3

12. **(5p)** If the QR factorization of a matrix A is given as

```
>> [Q,R]=qr(A)

Q =
   -0.2582    0.7115    0.6383    0.0468   -0.1321
   -0.5164   -0.5534    0.5003   -0.1403    0.3964
   -0.7746    0.1581   -0.5175   -0.2417   -0.2207
   -0.2582   -0.0791   -0.0863    0.9590    0.0016
         0    0.3953   -0.2588    0.0079    0.8813

R =
   -3.8730   -0.7746   -1.2910
         0    2.5298    1.5811
         0         0    2.4152
         0         0         0
         0         0         0
```

write down the system that must be solved to get the least squares solution to $Ax = b$ with $b = [1, 0, 0, 0, 1]^{\mathrm{T}}$. How many operations are (approximately) needed to solve the system? Do not solve.

Solution:

$$\begin{pmatrix} -3.8730 & -0.7746 & -1.2910 \\ 0 & 2.5298 & 1.5811 \\ 0 & 0 & 2.4152 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$

$$\begin{pmatrix} -0.2582 & -0.5164 & -0.7746 & -0.2582 & 0 \\ 0.7115 & -0.5534 & 0.1581 & -0.0791 & 0.3953 \\ 0.6383 & 0.5003 & -0.5175 & -0.0863 & -0.2588 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.2582 \\ 1.1068 \\ 0.3795 \end{pmatrix}$$

It is a 3×3 triangular system, requiring about $n^2 = 9$ operations.

13. **(5p)** When I calculate the eigenvalues and the singular values of a certain 5×5 matrix, I obtain the following results:

```
>> eigenvalues = eig(A)

eigenvalues =
   -1.2836
   -0.6084
   -0.0100
    0.1024
    4.1611

>> singular_values = svd(A)

singular_values =
    4.1611
    1.2836
    0.6084
    0.1024
```

```
0.0100
```

(a) Is the matrix invertible? Why?
Solution: Yes, because all its eigenvalues are different from 0.

(b) What kind of structure (of A) does this result suggest?
Solution: As the singular values are the absolute values of the eigenvalues, it suggests a symmetric matrix.

(c) Why does this result guarantee a basis of $\mathbb{R}^5$ consisting of eigenvectors of A?
Solution: Because the eigenvectors of a symmetric matrix associated to different eigenvalues are orthogonal.

(d) What eigenvalue would I get if I applied the inverse power method?
Solution: The smallest eigenvalue in modulus, -0.0100.

(e) If I had a matrix B with eigenvalues $\{-2.2836, -0.6084, -0.0100, 0.1024, 4.1611\}$, for which of the two matrices would the power method converge faster to 4.1611? Why?
Solution: It would converge faster for A because the ratio $|\lambda_2/\lambda_1|$ is smaller: $A(\lambda_2/\lambda_1) = 1.2836/4.1611$ and $B(\lambda_2/\lambda_1) = 2.2836/4.1611$

14. **(5p)** Given that

```
>> [U,S,V]=svd(A)

U =
   -0.5544   -0.7279    0.3310   -0.2305   -0.0000
    0.7496   -0.6132   -0.1982   -0.1509    0.0000
   -0.3504   -0.1358   -0.8951   -0.0136   -0.2397
   -0.0425    0.1808   -0.1734   -0.7178    0.6482
   -0.0781   -0.2072   -0.1412    0.6392    0.7228

S =
    4.1611         0         0         0         0
         0    1.2836         0         0         0
         0         0    0.6084         0         0
         0         0         0    0.1024         0
         0         0         0         0    0.0000

V =
   -0.5544    0.7279   -0.3310   -0.2305         0
    0.7496    0.6132    0.1982   -0.1509   -0.0000
   -0.3504    0.1358    0.8951   -0.0136    0.2397
   -0.0425   -0.1808    0.1734   -0.7178   -0.6482
   -0.0781    0.2072    0.1412    0.6392   -0.7228
```

How can you achieve a lossy compression of A of at least 50%?
Solution: A is a 5×5 matrix, so it has 25 entries. If we compress A with

its best rank-1 approximation,

$$4.1611 \begin{pmatrix} -0.5544 \\ 0.7496 \\ -0.3504 \\ -0.0425 \\ -0.0781 \end{pmatrix} \begin{pmatrix} -0.5544 & 0.7496 & -0.3504 & -0.0425 & -0.0781 \end{pmatrix}$$

the entries are at most 11. This is a 56% compression.

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