

Numerical Analysis — FMN011 — 130603

The exam lasts 4 hours. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

- (4p)** The fixed point iteration converges for both $g_1(x) = \cos(x)$ (to $x=0.7391$) and $g_2(x) = \cos^2(x)$ (to $x=0.6417$), but while the iteration converges to the fixed point of g_1 with 4 correct decimal digits in 23 iterations, it needs 206 iterations to achieve the same accuracy for g_2 . Explain why fixed point iteration converges in both cases, and why it converges faster for g_1 .
- (6p)** Consider Newton-Raphson's method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$
 - What is the convergence rate for Newton-Raphson's method for finding the root $x = 2$ of each of the following equations?
 - $f(x) = (x - 1)(x - 2)^2$
 - $f(x) = (x - 1)^2(x - 2)$
 - How should the formula be modified for multiple roots and what will be its rate of convergence?
 - Write out explicitly the formula for the Newton-Raphson iteration you would use for solving $(x - 1)(x - 2)^2 = 0$
- (4p)** Describe how the LU factorization with pivoting is used to solve $Ax = b$, and mention the computational complexity of each main step.
- (6p)** Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1.0001 & 1 \end{pmatrix}$$

- Find the infinity-norm condition number of A .
 - Find the error and the residual for the approximate solution $\begin{pmatrix} -1 & 3 \end{pmatrix}^T$ when $b = \begin{pmatrix} 2 & 2.0001 \end{pmatrix}^T$
 - Find a vector x satisfying $\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty$
- (6p)** The system $Ax = b$ was solved using the Gauss-Seidel iterative method, whose iteration matrix has the form $-(L + D)^{-1}U$. Matrix A is a strictly diagonally dominant $10^5 \times 10^5$ matrix that has 4×10^5 non-zero entries (4 entries per row).

- (b) If M is an orthogonal matrix, then $\|x\|_p = \|Mx\|_p$ for all positive integers p .
- (c) A Householder reflector is a matrix that is both symmetric and orthogonal.
10. **(4p)** Given a general square matrix A , how would you compute the following?
- (a) The smallest eigenvalue of A (in magnitude) and its corresponding eigenvector
- (b) The largest eigenvalue of A (in magnitude) and its corresponding eigenvector
- (c) The eigenvalue of A closest to some specified scalar β
- (d) All of the eigenvalues and eigenvectors of A

11. **(5p)** The following computations were done in Matlab:

```
>> A=[10 -12 -6;5 -5 -4; -1 0 3];
>> [U,S,V]=svd(A)
U =
   -0.8966   -0.2617    0.3574
   -0.4324    0.3420   -0.8343
    0.0961   -0.9025   -0.4198
S =
   18.6448         0         0
         0    2.8850         0
         0         0    0.2231
V =
   -0.6020   -0.0016   -0.7985
    0.6930    0.4958   -0.5234
    0.3967   -0.8684   -0.2974
```

How would you compute the rank-1 approximation of A ?

Hint: $A = \sum_{i=1}^r s_i u_i v_i^T$

12. **(5p)** Explain or illustrate how you can plot the trigonometric polynomial

$$P(t) = \frac{a_0}{\sqrt{8}} + \frac{2}{\sqrt{8}} \sum_{k=1}^3 \left(a_k \cos \frac{2\pi kt}{8} - b_k \sin \frac{2\pi kt}{8} \right) + \frac{a_4}{8} \cos \pi t,$$

that interpolates the points $(j, x_j), j = 0, \dots, 7; x_j \in \mathbb{R}$, using the discrete Fourier transform.

13. **(5p)** Describe how quantization together with the discrete cosine transform is used for image compression.

14. **(5p)** The Shannon information is $I = -\sum_{i=1}^k p_i \log_2 p_i$. Draw a Huffman tree and convert the message

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to a bit stream, using Huffman coding. What is the average number of bits needed per symbol?