**Solution — Numerical Analysis — FMN011 — 100528**

The exam lasts 4 hours and has 13 questions. A minimum of 35 points out of
the total 70 are required to get a passing grade. These points will be added to
those you obtained in your two home assignments, and the final grade is based
on your total score.

**Justify all your answers and write down all important steps**. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture
notes or any other electronic or written material.

1. **(6p)** Suppose a numerical method is used to solve the fixed point problem
   $x = g(x)$, where $g : \mathbb{R} \to \mathbb{R}$ and the computed value of $x$ is $\hat{x}$. Suppose
   the exact value of the solution is $x^*$.

   (a) Write the formula for the absolute error of the approximation.
       ***Solution:*** $|x^* - \hat{x}|$
   (b) Write the formula for the relative error of the approximation.
       ***Solution:*** $|x^* - \hat{x}|/|x^*|$
   (c) Write the formula for the residual or backward error.
       ***Solution:*** $|g(x^*) - x^*|$

2. **(4p)** If a function $f : \mathbb{R} \to \mathbb{R}$ is approximated by its degree 3 Taylor
   polynomial centered at 0, write the formula for the truncation error.

   ***Solution:*** $\dfrac{f^{(4)}(\zeta)}{4!}x^4$, $\zeta$ *is between 0 and* $x$.

3. **(5p)** Fill in the blanks (marked `-->`) to complete this implementation of
   the bisection method. What mathematical problem does it solve?

```
    function [c,max_possible_err,res] = bisection(f,a,b,tol)
    %  f(a), f(b) must have opposite signs
    % x is the approximate solution
--> while (b-a)/2 > tol
-->     c = (a + b)/2;
        if f(c)*f(a) > 0
-->         a = c;
        elseif f(c)*f(b) > 0
-->         b = c;
        else break
        end
    end
--> max_possible_err = (b - a)/2;
    res = f(c);
```

4. **(4p)** Illustrate three iteration steps of the Newton-Raphson method

$$p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}$$

applied to the function displayed on then following page, with starting point at $x_0 = 3$. Clearly mark each of the iterates $x_j$. You may detach this page, work out the result and hand it in with your answers.

*Solution:*

**Illustration of the first 2 iteration steps of the Newton-Raphson method applied to the function plotted below, starting at $x_0 = 3$**
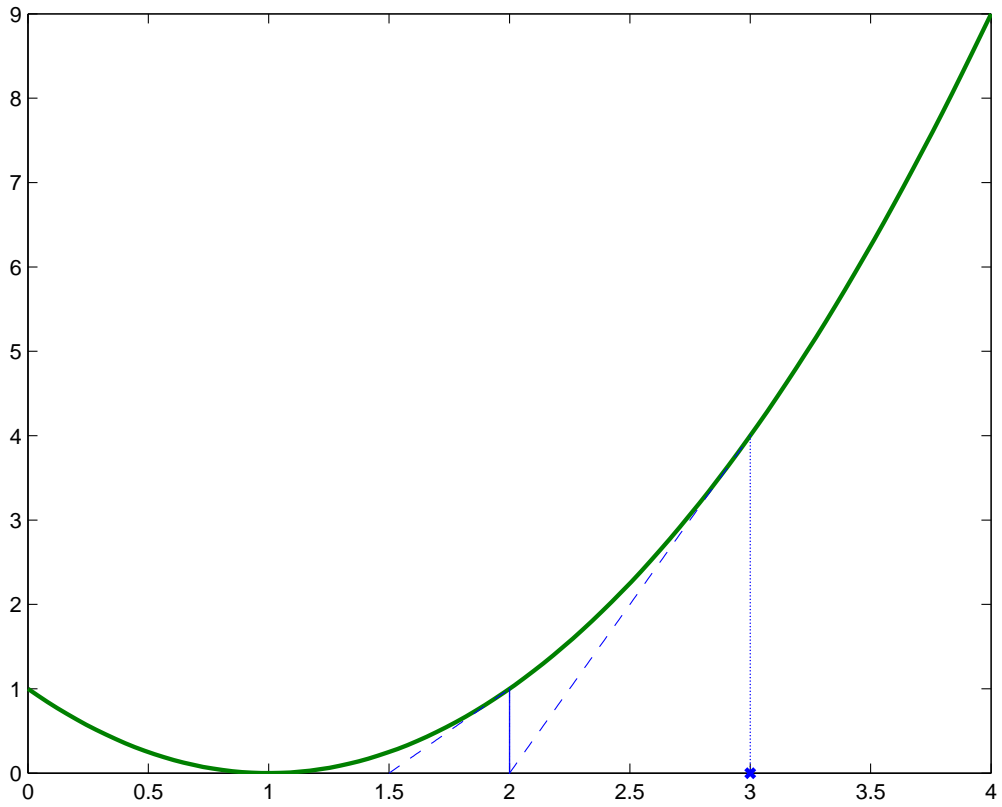
5. The following Matlab code does Gauss elimination on a matrix A.

```
for j=1:n-1
    for i=j+1:n
        m=a(i,j)/a(j,j);
        b(i)=b(i)-m*b(j);
        for k=j+1:n
            a(i,k)=a(i,k)-m*a(j,k);
        end
    end
end
```

(a) **(3p)** Modify the code to optimize it for tridiagonal matrices, i.e. with structure

$$\begin{pmatrix} X & X & & & & \\ X & X & X & & & \\ & \ddots & \ddots & \ddots & & \\ & & X & X & X \\ & & & X & X \end{pmatrix}$$

*Solution:*

```
for j=1:n-1
    m=a(j+1,j)/a(j,j);
    b(j+1)=b(j+1)-m*b(j);
    a(j+1,j+1)=a(j+1,j+1)-m*a(j,j+1);
end
```

(b) **(3p)** Count the number of arithmetic operations (additions, subtractions, multiplications, divisions and square roots) needed to solve a system of $n$ equations with a tridiagonal matrix. If needed, you may use the formulas

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

*Solution:* $5(n-1)$

6. **(5p)** Specify an elementary matrix that zeros out the last 3 components of vector $[1\ 2\ 3\ 4\ 5]^T$. *Solution:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 0 \\ 0 & 0 & -4/3 & 1 & 0 \\ 0 & 0 & 0 & -5/4 & 1 \end{pmatrix}$$

7. (a) **(2p)** Why is interpolation by a polynomial of high degree often unsatisfactory?

*Solution: Because a polynomial of high degree may exhibit too much "wiggling".*

(b) **(2p)** How should the interpolation points be placed in an interval in order to guarantee convergence of the polynomial interpolation polynomial to sufficiently smooth functions on the interval as the number of points increases?

*Solution: They should be the projection of points equidistantly spaced on the half unit circle. These are the Chebyshev nodes.*

(c) **(2p)** For uniformly spaced points, is the error in a Lagrange interpolation greater around the middle of the range or near the endpoints?

*Solution: Near the endpoints.*

(d) **(2p)** How many roots does the Chebyshev polynomial of order $n$ has in $(-\infty, \infty)$? In $[-1, 1]$?

*Solution: $n$ in both intervals.*

(e) **(2p)** A linear change of interval between [-1, 1] and [a, b] can be accomplished by the formula

$$x = \frac{b-a}{2}t + \frac{a+b}{2}.$$

If the zeros of the $n$-th degree Chebychev polynomial are

$$\cos\frac{(2k-1)\pi}{2n},$$

which points should you use to do a Chebyshev interpolation in the interval $[0, 4]$?

*Solution:* $2\cos\dfrac{(2k-1)\pi}{2n} + 2$

8. **(5p)** Describe the relation between the discrete Fourier transform $y = F_n x$ and the polynomial

$$P_n(t) = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}\left(a_k\cos\frac{2\pi k(t-c)}{d-c} - b_k\sin\frac{2\pi k(t-c)}{d-c}\right), \quad t \in [c, d].$$

*Solution: $P_n(t_j) = x_j$ for $t_j$ equally spaced in $[c, d]$ and $F_n x_j = a_j + b_j i$*

9. **(5p)** The first 9 components of the discrete Fourier transform of a vector $x \in \mathbb{R}^{16}$ are $136, -8 + 40.219i, -8 + 19.314i, -8 + 11.973i, -8 + 8i, -8 + 5.3454i, -8+3.3137i, -8+1.5913i, -8$. What are the missing components?

*Solution: $-8-1.5913i, -8-3.3137i-8-5.3454i, -8-8i, -8-11.973i, -8-19.314i, -8-40.219i$*

10. (a) **(2p)** The Shannon information formula is

$$I = -\sum_{i=1}^{k}p_i\log_2 p_i$$

Calculate the average least number of bits needed to code the matrix

$$A = \begin{pmatrix} 4 & 7 & 7 \\ 9 & 0 & 8 \\ 8 & 8 & 7 \end{pmatrix}$$

*Solution: For $x = [0\ 4\ 7\ 8\ 9]$, $p = [1\ 1\ 3\ 3\ 1]/9$ and $I = 2.1133$, so the average number of symbols for the matrix is $19.02 \approx 19$.*

(b) **(2p)** Construct the Huffman code for $A$.

**Solution:** *The corresponding codes are* $(0, 110), (4, 1110), (7, 10), (8, 0), (9, 1111)$.
*The Huffman code for $A$ is*

$$A = \begin{pmatrix} 1110 & 10 & 10 \\ 1111 & 110 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

(c) **(2p)** What is the average for this coding? What is the average if the standard binary system is used?

**Solution:** $20/9 = 2.2222$ *and* $29/9 = 3.2222$

11. **(5p)** True or false (support your answer with a short explanation or counterexample):

(a) The DFT implies a periodic extension of the function defined over a finite interval, and the DCT implies an even extension of the function.
**Solution:** *True. The periodic extension of DCT is even because it only consists of cosine functions.*

(b) The discrete cosine transform is a linear transformation that is not necessarily invertible. **Solution:** *False, because the transformation matrix is orthogonal.*

(c) The order of complexity of the DCT algorithm is the same as that of the DFT algorithm. **Solution:** *True. The DCT is basically the real counterpart of the DFT.*

(d) Quantization and Huffman coding are examples of lossy compression.
**Solution:** *Quantization is lossy but HUffman is lossless.*

(e) In the SVD of matrix $A$, the columns of $U$ are orthonormal eigenvectors of $AA^T$. **Solution:** *True. $AA^T u = s^2 u$*

12. **(4p)** The $QR$ factorization of matrix $A$ is

$$\begin{pmatrix} 0.24759 & 0.51408 & 0.67406 & -0.4691 \\ 0.55709 & -0.76621 & 0.29789 & -0.11762 \\ 0.49519 & 0.29563 & 0.15133 & 0.8028 \\ 0.61898 & 0.24745 & -0.65879 & -0.34875 \end{pmatrix} \begin{pmatrix} 16.155 & 11.266 & 12.132 & 6.3136 \\ 0 & 8.1907 & 0.52812 & 4.1356 \\ 0 & 0 & 5.5257 & 6.3735 \\ 0 & 0 & 0 & 2.1007 \end{pmatrix}$$

Can you give the eigenvectors of $A$ from this information?

**Solution:** *No, to get the eigenvalues of $A$ we must use the QR algorithm. This is the QR factorization.*

13. **(5p)** The $QR$ algorithm for matrix $A$ is

$$\begin{aligned} A_0 &\equiv A = Q_1 R_1 \\ A_1 &\equiv R_1 Q_1 = Q_2 R_2 \\ A_2 &\equiv R_2 Q_2 = Q_3 R_3 \\ &\vdots \\ A_k &\equiv R_k Q_k \end{aligned}$$

If $A$ is a symmetric matrix with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, where do we find the eigenvalues of $A$? How do we find the eigenvectors of $A$?

**Solution:** *As $k \to \infty$ the matrix $A_k$ converges to a diagonal matrix whose diagonal entries are the eigenvalues of $A$. The product $Q_1 Q_2 \cdots Q_k$ converges to an orthogonal matrix whose columns are the eigenvectors of $A$.*

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