Numerical Analysis — FMN011 — 100528

The exam lasts 4 hours and has 13 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(6p)** Suppose a numerical method is used to solve the fixed point problem $x = g(x)$, where $g : \mathbb{R} \to \mathbb{R}$ and the computed value of x is $\hat{x}$. Suppose the exact value of the solution is x^*.
   (a) Write the formula for the absolute error of the approximation.
   (b) Write the formula for the relative error of the approximation.
   (c) Write the formula for the residual or backward error.

2. **(4p)** If a function $f : \mathbb{R} \to \mathbb{R}$ is approximated by its degree 3 Taylor polynomial centered at 0, write the formula for the truncation error.

3. **(5p)** Fill in the blanks (marked `-->`) to complete this implementation of the bisection method. What mathematical problem does it solve?

```
    function [c,max_possible_err,res] = bisection(f,a,b,tol)
    %  f(a), f(b) must have opposite signs
    %  c is the approximate solution
--> while (b-a)/2 >
-->    c =
       if f(c)*f(a) > 0
-->        a =
       elseif f(c)*f(b) > 0
-->        b =
       else break
       end
    end
--> max_possible_err =
    res = f(c);
```

4. **(4p)** Illustrate three iteration steps of the Newton-Raphson method

$$p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}$$

applied to the function displayed on the last page (page 5) of this exam, with starting point at $x_0 = 3$. Clearly mark each of the iterates x_j. You may detach this page, work out the result and hand it in with your answers.

5. The following Matlab code does Gauss elimination on a system $Ax = b$.

```
for j=1:n-1
   for i=j+1:n
      m=a(i,j)/a(j,j);
      b(i)=b(i)-m*b(j);
      for k=j+1:n
         a(i,k)=a(i,k)-m*a(j,k);
      end
   end
end
```

(a) **(3p)** Modify the code to optimize it for tridiagonal matrices, i.e. with structure

$$\begin{pmatrix} X & X & & & \\ X & X & X & & \\ & \ddots & \ddots & \ddots & \\ & & X & X & X \\ & & & X & X \end{pmatrix}$$

(b) **(3p)** Count the number of arithmetic operations (additions, subtractions, multiplications, divisions and square roots) needed to solve a system of n equations with a tridiagonal matrix. You may use the formulas

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \quad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

6. **(5p)** Specify an elementary matrix that zeros out the last 3 components of vector $[1\ 2\ 3\ 4\ 5]^T$.

7. (a) **(2p)** Why is interpolation by a polynomial of high degree often unsatisfactory?

(b) **(2p)** To guarantee convergence of a polynomial interpolation to sufficiently smooth functions as the number of points increases on an interval, how should the interpolation points be placed?

(c) **(2p)** For uniformly spaced points, is the error in a Lagrange interpolation greater around the middle of the interval or near the endpoints?

(d) **(2p)** How many roots does the Chebyshev polynomial of order n have in $(-\infty, \infty)$? In $[-1, 1]$?

(e) **(2p)** A linear change of interval between [-1, 1] and [a, b] can be accomplished by the formula

$$x = \frac{b-a}{2}t + \frac{a+b}{2}.$$

If the zeros of the n-th degree Chebychev polynomial are

$$x_k = \cos\frac{(2k-1)\pi}{2n},$$

which points should you use as x-data points to do a Chebyshev interpolation in the interval $[0, 4]$?

8. **(5p)** Describe the relation between the discrete Fourier transform $y = F_n x$ and the polynomial

$$P_n(t) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(a_k \cos \frac{2\pi k(t-c)}{d-c} - b_k \sin \frac{2\pi k(t-c)}{d-c} \right), \quad t \in [c, d].$$

9. **(5p)** The first 9 components of the discrete Fourier transform of a vector $x \in \mathbb{R}^{16}$ are $136, -8+40.219i, -8+19.314i, -8+11.973i, -8+8i, -8+5.3454i, -8+3.3137i, -8+1.5913i, -8$. What are the missing components?

10. (a) **(2p)** The Shannon information formula is

$$I = -\sum_{i=1}^{k} p_i \log_2 p_i$$

Calculate the average least number of bits needed to code the matrix

$$A = \begin{pmatrix} 4 & 7 & 7 \\ 9 & 0 & 8 \\ 8 & 8 & 7 \end{pmatrix}$$

(b) **(2p)** Construct the Huffman code for A.

(c) **(2p)** What is the average for this coding? What is the average if the standard binary system is used for the matrix entries?

11. **(5p)** True or false (support your answer with a short explanation or counterexample):

(a) The DFT implies a periodic extension of the function defined over a finite interval, and the DCT implies an even extension of the function.

(b) The discrete cosine transform is a linear transformation that is not necessarily invertible.

(c) The order of complexity of the DCT algorithm is the same as that of the DFT algorithm.

(d) Quantization and Huffman coding are examples of lossy compression.

(e) In the SVD of matrix A, the columns of U are orthonormal eigenvectors of AA^T.

12. **(4p)** The QR factorization of matrix A is

$$\begin{pmatrix} 0.24759 & 0.51408 & 0.67406 & -0.4691 \\ 0.55709 & -0.76621 & 0.29789 & -0.11762 \\ 0.49519 & 0.29563 & 0.15133 & 0.8028 \\ 0.61898 & 0.24745 & -0.65879 & -0.34875 \end{pmatrix} \begin{pmatrix} 16.155 & 11.266 & 12.132 & 6.3136 \\ 0 & 8.1907 & 0.52812 & 4.1356 \\ 0 & 0 & 5.5257 & 6.3735 \\ 0 & 0 & 0 & 2.1007 \end{pmatrix}$$

Can you give the eigenvectors of A from this information?

13. **(5p)** The QR algorithm for matrix A is

$$\begin{aligned} A_0 &\equiv A = Q_1 R_1 \\ A_1 &\equiv R_1 Q_1 = Q_2 R_2 \\ A_2 &\equiv R_2 Q_2 = Q_3 R_3 \\ &\vdots \\ A_k &\equiv R_k Q_k \end{aligned}$$

If A is a symmetric matrix with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$, where do we find the eigenvalues of A? How do we find the eigenvectors of A?

Illustration of the first 3 iteration steps of the Newton-Raphson method applied to the function plotted below, starting at $x_0 = 3$