Tenta Numerical Analysis för D, L. FMN011, 090527

This exam starts at 8:00 and ends at 12:00. To get a passing grade for the course you need 35 points in this exam and an accumulated total (this exam plus the two computational exams) of 50 points.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(5p)** Name one numerical method that is suitable for solving the following problems:

   (a) $x - e^x = 0$

   (b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

   (c) Find a fourth degree polynomial passing through 5 given points.

   (d) Construct a smooth curve passing through 50 given points.

   (e) Solve the initial value problem $y' = y^2/t + \sin y$, $y(1) = 1$ in the interval $[1, 5]$.

2. **(5p)** Consider the three methods

   (a) $x_{n+1} = f(x_n)$

   (b) $p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}$

   (c) $p_k = p_{k-1} - \frac{f(p_{k-1})(p_{k-1} - p_{k-2})}{f(p_{k-1}) - f(p_{k-2})}$

   State (all) those that have each of the following properties:

   (a) Solves $f(x) = 0$.

   (b) Usually has quadratic convergence.

   (c) Will converge only if $|f(x)| < 1$.

   (d) Is an iteration method.

   (e) Can be used to find a root of $16x^2 - 32x + 15 = 0$, with $f(x) = \sqrt{32x - 15}/4$.

3. **(5p)** Suppose you have a square linear system of equations $Ax = b$. Name the three elementary row operations that yield an equivalent system.

4. **(5p)** The following statement is wrong. Correct the statement.

   The iterative method
   $$x_{k+1} = Bx_k + c$$
   is convergent (for any initial guess x_0) only if $\|B\|_p < 1$ for some p.

5. **(5p)** Write the polynomial passing through the points (0, 1), (-2.5, 7.2) and (3.1, -2), if the Lagrange polynomials have been computed and are $x(x+2.5)/17.36$, $x(x-3.1)/14$ and $(x+2.5)(3.1-x)/7.75$.

6. **(5p)** Consider the Bézier curve with control points (2,3), (1,-1), (4,1) and (5,0). Remembering that the cubic Bernstein polynomials are $(1-t)^3, 3(1-t)^2t, 3(1-t)t^2, t^3$, write down the parametric equations of the curve, and give the first control point of another cubic Bézier curve that starts at (5,0) and ends at (6,-2), and joins the previous curve as smoothly as possible.

7. **(5p)** The following table is found on an oatmeal porridge package:

Portions	Oats	Water
1	1	2.5
2	2	4.5
4	4	9

   Determine the straight line that passes through (0, 0) and best fits the given data.

8. **(5p)** Write an algorithm that solves the least squares problem $Ax = b$, where A is an 25×3 matrix, using QR factorization.

9. **(5p)** Consider the formula
   $$f''(t) = \frac{f(t+2h) - 2f(t+h) + f(t)}{h^2} - hf'''(\zeta)$$
   Suppose that the rounding errors are such that
   $$\hat{f}(t) \approx f(t) + \epsilon$$
   Calculate the total error function $E(h)$, such that $|f''(t) - \hat{f}''(t)| \leq E(h)$.

10. **(5p)** Describe how the approximation

$$\int_a^b f(x)dx \approx S(a,c) + S(c,b) + \frac{1}{15}[S(a,c) + S(c,b) - S(a,b)]$$

can be used to construct an adaptive Simpson quadrature algorithm.

11. **(5p)** When an initial value problem was solved with a certain Runge-Kutta method using different step-sizes, the following errors were observed:

h	norm-1 error/N
1/1000	7.8013×10^{-9}
1/2000	1.9499×10^{-9}
1/4000	4.8743×10^{-10}
1/8000	1.2185×10^{-10}

What is the order of convergence of this method?

12. **(5p)** Write the MATLAB `function yprime=myode(t,y)` you would need to solve the differential equation $y''' = yt - 3y'$.

13. **(5p)** Study the following algorithm:

$$\begin{aligned} y_{k-1} &= \frac{x_{k-1}}{\|x_{k-1}\|_2} \\ x_k &= Ay_{k-1} \\ \lambda_k &= y_{k-1}^T x_k \end{aligned}$$

What method is this and what does it compute?

14. **(5p)** Suppose a 128×128 matrix A has a singular value decomposition $A = USV^T$, and the first two singular values s_1 and s_2 are much larger than the rest of the singular values. How can one compress the main part of the information by using the SVD

$$A = \sum_{i=1}^{n} s_i u_i v_i^T$$

and what would the compression ratio be?

Lycka till!
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