Lösning till Tentamen Numerical Analysis D3, FMN011, 070528

The exam starts at 8:00 and ends at 12:00. To get a passing grade for the course you need 35 points in this exam and an accumulated total (this exam plus the two computational exams) of 50 points.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(5p)** Verify that the equation $\cos x - x = 0$ has a root on the interval $(0, 1)$. Use the bisection method to approximate the root so that the absolute error is less than 0.05.

   Solution: $f(x) = \cos x - x$ is continuous, $f(0) > 1$ and $f(1) < 0$, so there is a $c \in (0, 1)$ such that $f(c) = 0$.
   If p_n is the n-th iterate when applying the bisection method in the interval $[0, 1]$, then $|p_n - c| \leq \frac{(1-0)}{2^n} \Rightarrow \frac{1}{2^n} < 0.05 \Rightarrow n \geq 5$.

a	b	p	$f(p)$
0	1	0.5000	0.3776
0.5	1	0.7500	-0.0183
0.5	0.75	0.6250	0.1860
0.625	0.75	0.6875	0.0853
0.6875	0.75	0.7188	0.0339

   $c \approx 0.7188$.

2. **(5p)** Perform 3 iterations of the secant method to determine an approximation to the root of $\cos x - x = 0$ on the interval $(0, 1)$.

   Solution: The secant method uses an approximation to the derivative in Newton's method: $x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$. It needs two starting values, a and b.

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 1 \\
x_3 &= 0.6851 \\
x_4 &= 0.7363 \\
x_5 &= 0.7391
\end{aligned}
$$

$c \approx 0.7391$

3. Given $Ax = b$, with A invertible, what effect on the solution vector x results from

   (a) **(2p)** Permuting the rows of A and b in the same manner

   (b) **(2p)** Permuting only the columns of A

   Solution: (a) The equations are written in a different order, but the solution is the same. (b) The elements of the solution vector are permuted in the same manner.

4. **(5p)** Consider the linear system

$$\begin{aligned} x - 6y - 9z &= 5 \\ 4x + 2y + z &= 12 \\ 7y - 2z &= 9 \end{aligned}$$

   Write it in a matrix-vector form that ensures convergence of the Gauss-Seidel method (justify), and calculate one iteration with the starting values $x = y = z = 0$.

   Solution: Convergence is ensured if the matrix is strictly diagonally dominant.

$$\begin{pmatrix} 4 & 2 & 1 \\ 0 & 7 & -2 \\ 1 & -6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{aligned} x_{i+1} &= \frac{1}{4}(12 - 2y_i - z_i) \\ y_{i+1} &= \frac{1}{7}(9 + 2z_i) \\ z_{i+1} &= -\frac{1}{9}(5 - x_{i+1} + 6y_{i+1}) \end{aligned}$$

   $x_1 = 3, y_1 = 9/7 = 1.2857, z_1 = -68/63 = -1.079.$

5. Assume that the polynomial $P_9(x)$ interpolates the function $f(x) = e^{-2x}$ at the 10 evenly spaced points $0, 1/9, 2/9, \ldots, 8/9, 1$.

   (a) **(3p)** Find an upper bound for the error $|f(1/2) - P_9(1/2)|$.

   (b) **(3p)** How many decimal places can you guarantee to be correct if $P_9(1/2)$ is used to approximate $1/e$?

Solution: (a) We use the formula for 10 data points.

$$\begin{aligned}
|f(x) - P_9(x)| &= \frac{x(x-\frac{1}{9})(x-\frac{2}{9})\cdots(x-1)}{10!} f^{(10)}(c) \\
|f(1/2) - P_9(1/2)| &= \frac{|\frac{1}{2}(\frac{1}{2}-\frac{1}{9})(\frac{1}{2}-\frac{2}{9})\cdots(\frac{1}{2}-1)|}{10!} 2^{10} e^{-2c}, \quad 0 \le c \le 1 \\
&= 0.7058 \times 10^{-10} e^{-2c}, \quad 0 \le c \le 1 \\
&\le 0.7058 \times 10^{-10}
\end{aligned}$$

(b)

$$P_9(1/2) = f(1/2) \pm 0.7058 \times 10^{-10}$$

We can guarantee up to 9 decimal places.

6. **(5p)** How many natural cubic splines on $[0,2]$ are there for the given data $(0,0), (1,1), (2,2)$? Exhibit one such spline.

   Solution: There is a unique natural cubic spline passing though a set of points with different abscissas. The three given points lie on a straight line, $y = x$, and this line satisfies the definition of a natural cubic spline.

7. **(5p)** Does the over-determined system

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \approx \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

   have a unique least squares solution for every set $b = (b_{11}, b_{21}, b_{31})^T$? Do not solve. Justify your answer.

   Solution: There is no unique solution because the matrix is not full-rank.

8. **(5p)** We know that

$$A = \begin{pmatrix} 4 & 4 \\ 2 & -7 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & -2/3 \\ -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ 0 & 9 \end{pmatrix}$$

   Note that $(2/3, 1/3, -2/3)$ and $(2/3, -2/3, 1/3)$ are mutually orthogonal. Compute the QR factorization of A.

Solution: We add a third independent column vector $u_3 = (1, 0, 0)^T$, and orthogonalize it (Gram-Schmidt):

$$
\begin{aligned}
v_3 &= u_3 - \frac{v_1^T u_3}{v_1^T v_1} v_1 - \frac{v_2^T u_3}{v_2^T v_2} v_2 \\
&= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix} - \begin{pmatrix} 2/3 & -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \\
&= \begin{pmatrix} 1/9 \\ 2/9 \\ 2/9 \end{pmatrix}
\end{aligned}
$$

Normalize the vector,

$$\frac{v_3}{\|v_3\|_2} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ 0 & 9 \\ 0 & 0 \end{pmatrix}$$

9. **(5p)** A given 3×3 matrix M has three different eigenvalues in the interval $[a, b]$. Modify the following Matlab code so that it calculates the eigenvalue of M closest to a given $s \in [a, b]$.

```
function lam=power(M,x,k)
for i=1:k
    u=x/norm(x);
    x=M*u;
    lam=u'*x;
end
```

Solution: Use the shifted inverse method, $(M - sI)x_{i+1} = x_i$.

```
function lam = sipower(M,x,k,s)
A = M-s*eye(size(A));
for i = 1:k
    u = x/norm(x);
    x = A\u;
    lam = u'*x;
end
lam = 1/lam + s;
```

10. **(5p)** If

$$A = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}$$

what is the best rank-1 approximation to A?

Solution: If the singular value decomposition of a matrix is $A = USV^T$, its best rank-1 approximation is

$$s_1 u_1 v_1^T = 2 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 2/5 & 4/5 \\ 4/5 & 8/5 \end{pmatrix}$$

11. **(5p)** Develop a formula for a two-point **backward**-difference formula for approximating $f'(x)$, including the error term.

Solution: For a two-point formula we need $f(x_{i-1})$.

$$\begin{aligned} f(x-h) &= f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) + \mathcal{O}(h^3) \\ f(x) &= f(x) \end{aligned}$$

Subtracting the first equation from the second one,

$$\begin{aligned} f(x) - f(x-h) &= hf'(x) - \frac{1}{2}h^2 f''(x) + \mathcal{O}(h^3) \\ f'(x) &= \frac{f(x) - f(x-h)}{h} + \frac{1}{2}hf''(c) \end{aligned}$$

12. **(5p)** Develop a composite version of the quadrature rule $\int_{x_0}^{x_4} f(x)dx \approx \frac{4}{3}h[2f(x_1) - f(x_2) + 2f(x_3)]$.

Solution:

$$\int_{x_0}^{x_{4N}} f(x)dx = \int_{x_0}^{x_4} f(x)dx + \int_{x_4}^{x_8} f(x)dx + \cdots + \int_{x_{4N-4}}^{x_{4N}} f(x)dx$$

$$\approx \frac{4}{3}h[2f(x_1) - f(x_2) + 2f(x_3) + 2f(x_5) - f(x_6) + 2f(x_7) + \cdots]$$

$$\int_{x_0}^{x_{4N}} f(x)dx \approx \frac{4}{3}h \left[2\sum_{i=1}^{2N} f(x_{2i-1}) - \sum_{i=1}^{N} f(x_{4i-2}) \right]$$

13. **(5p)** Apply the implicit Euler method with $h = 0.1$ to the initial value problem $y' = ty^2$, $y(0) = 1$, and approximate the solution with 6 decimal digits at $t = 0.1$ by performing two Newton iterations with the initial value of the differential equation as starting value.

Solution:

$$\begin{aligned} f(t,y) &= ty^2 \\ v_0 &= 1 \\ t_1 &= 0.1 \\ v_1 &= v_0 + ht_1v_1^2 = 1 + 0.01v_1^2 \\ v_1 - 1 - 0.01v_1^2 &= 0 \end{aligned}$$

To solve this non-linear equation for v_1 use Newton's method:

$$\begin{aligned} g(u) &= u - 1 - 0.01u^2 \\ g'(u) &= 1 - 0.02u \\ u_{i+1} &= u_i - \frac{u_i - 1 - 0.01u_i^2}{1 - 0.02u_i} \\ u_0 &= 1 \\ u_1 &= 1 + \frac{0.01}{0.98} = \frac{99}{98} = 1.010204 \\ u_2 &= 1.010205 \end{aligned}$$

$$y(0.1) \approx v_1 \approx 1.010205$$

14. Suppose we want to construct an adaptive IVP solver of order 3, with an error tolerance TOL.

   (a) **(1p)** How can we get an estimate of the local error at each step?

   (b) **(1p)** What is the condition that must be satisfied at each step n?

   (c) **(3p)** How do we decide on the next stepsize, h_{n+1}, after the step is accepted?

Solution: (a) Suppose the approximate solution using the order 3 solver is u_n, and the approximate solution using an order 4 solver is v_n, then we can estimate the local error as $\hat{e}_n = |v_n - u_n|$. (b) $\hat{e}_n \le \mathrm{TOL}$. (c) A model for the error is $e_n = Ch_n^4$, so $C \approx \hat{e}_n/h_n^4$. We would like

e_{n+1} =TOL, or equivalently, Ch_{n+1}^4 =TOL; therefore we ask that

$$\begin{aligned}\hat{e}_n \frac{h_{n+1}^4}{h_n^4} &= \text{TOL} \\ h_{n+1}^4 &= \frac{\text{TOL}}{\hat{e}_n} h_n^4 \\ h_{n+1} &= \left(\frac{\text{TOL}}{\hat{e}_n}\right)^{1/4} h_n\end{aligned}$$

List of formulas:

1. Interpolation error: $f(x) - P(x) = \dfrac{\prod(x - x_i)}{n!} f^{(n)}(c)$.
2. Natural cubic spline: $S''(x_1) = 0, S''(x_{n-1}) = 0$.
3. Newton's method: $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$.
4. Projection of u onto v: $proj_v u = \dfrac{v^T u}{v^T v} v$

Lycka till!
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