Tentamen Numerical Analysis D3, FMN011, 070528

The exam starts at 8:00 and ends at 12:00. To get a passing grade for the course you need 35 points in this exam and an accumulated total (this exam plus the two computational exams) of 50 points.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. (a) **(2p)** Verify that the equation $\cos x - x = 0$ has a root on the interval $(0, 1)$.

   (b) **(3p)** Use the bisection method to approximate the root so that the absolute error is less than 0.05.

2. (a) **(2p)** Write out the formula for the secant method applied to this particular problem.

   (b) **(3p)** Use the secant method to determine an approximation to the root of $\cos x - x = 0$ on the interval $(0, 1)$, so that the residual is less than 0.005.

3. Given $Ax = b$, with A invertible, describe exactly what effect on the solution vector x results from

   (a) **(2p)** Permuting two rows of A and b in the same manner

   (b) **(2p)** Permuting only two columns of A

4. **(5p)** Consider the linear system

$$\begin{aligned} x - 6y - 9z &= 5 \\ 4x + 2y + z &= 12 \\ 7y - 2z &= 9 \end{aligned}$$

   Write it in a matrix-vector form that ensures convergence of the Gauss-Seidel method (justify), and calculate one iteration with the starting values $x = y = z = 0$.

5. Assume that the polynomial $P_9(x)$ interpolates the function $f(x) = e^{-2x}$ at the 10 evenly spaced points $0, 1/9, 2/9, \ldots, 8/9, 1$.

(a) **(3p)** Find an upper bound for the error $|f(1/2) - P_9(1/2)|$.

(b) **(3p)** How many decimal places can you guarantee to be correct if $P_9(1/2)$ is used to approximate $1/e$?

6. **(5p)** How many natural cubic splines on $[0, 2]$ are there for the given data $(0, 0), (1, 1), (2, 2)$? Exhibit one such spline.

7. **(5p)** Does the over-determined system

$$\begin{pmatrix} 1 & -2 \\ 1 & -2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \approx \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

have a unique least squares solution for every set $b = (b_{11}, b_{21}, b_{31})^T$? Do not solve. Justify your answer.

8. **(5p)** We know that

$$A = \begin{pmatrix} 4 & 4 \\ 2 & -7 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & -2/3 \\ -2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 6 & -3 \\ 0 & 9 \end{pmatrix}$$

Note that $(2/3, 1/3, -2/3)$ and $(2/3, -2/3, 1/3)$ are mutually orthogonal. Compute the QR factorization of A.

9. **(5p)** A given 3×3 matrix M has three different eigenvalues in the interval $[a, b]$. Modify the following Matlab code so that it calculates the eigenvalue of M closest to a given $s \in [a, b]$.

```
function lam=power(M,x,k)
for i=1:k
    u=x/norm(x);
    x=M*u;
    lam=u'*x;
end
```

10. **(5p)** If

$$A = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{pmatrix}$$

what is the best rank-1 approximation to A?

11. **(5p)** Develop a formula for a three-point **backward**-difference formula for approximating $f'(x)$, including the error term.

12. **(5p)** Develop a composite version of the open quadrature rule $\int_{x_0}^{x_4} f(x)dx \approx \frac{4}{3}h[2f(x_1) - f(x_2) + 2f(x_3)]$.

13. **(5p)** Apply the implicit Euler method with $h = 0.1$ to the initial value problem $y' = ty^2, \ y(0) = 1$, and approximate the solution with 6 decimal digits at $t = 0.1$ by performing two Newton iterations with the initial value of the differential equation as starting value.

14. Suppose we want to construct an adaptive IVP solver of order 3, with an error tolerance TOL.

    (a) **(1p)** How can we get an estimate of the local error at each step?

    (b) **(1p)** What is the condition that must be satisfied at each step n?

    (c) **(3p)** How do we decide on the next stepsize, h_{n+1}, after the step is accepted?

List of formulas:

1. Interpolation error: $f(x) - P(x) = \frac{\prod(x - x_i)}{n!} f^{(n)}(c)$.
2. Natural cubic spline: The second derivatives at the endpoints are zero.
3. Newton's method: $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$.
4. Projection of u onto v: $proj_v u = \frac{v^T u}{v^T v} v$

Lycka till!
C.Arévalo